Indian Statistical Institute
M. Math. II Year
Semestral Examination 2010-2011

Date: 26-11-2010 Fourier Analysis

Max Marks you can get is 50.

Notations: 1) For f in $L^1(R^n)$ the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int dx\ e^{-i\xi x}\ f(x).$$

2) If for some reason you have to omit the factor $(2\pi)^{-n/2}$ you have to mention it.

1. Let $f \in L^1[-\pi, \pi]$ and have period 2π. Define $S_n(f, x)$, $\sigma_n(f, x)$ by

$$S_n(f, x) = \sum_{k=-n}^{n} \hat{f}(k)\ e_k(x)$$

where $\hat{f}(k) = \langle f, e_k \rangle$, $e_k(x) = \frac{e^{+ikx}}{\sqrt{2\pi}}$

$$\sigma_n(f, x) = \frac{1}{n+1} \sum_{j=0}^{n} S_j(f, x).$$

(a) Show that $\sum_{k=-n}^{n} e^{ik\theta} = D_n(\theta) = \sin[(n + \frac{1}{2})\theta] / \sin[\frac{1}{2}\theta]$. (1)

(b) Prove that $\sum_{0}^{n} D_j(\theta) = \frac{\sin^2[(\frac{n+1}{2})\theta]}{\sin^2(\frac{\theta}{2})}$. (1)

(c) Prove Riemann localisation lemma viz. $\int_{\pi \geq |t| > \delta} dt\ f(t)\ D_n(t) \to 0$ as $n \to \infty$ for each $\delta > 0$ and each f in $L^1[-\pi, \pi]$. (1)

(d) State and (e) Prove Dini's condition for the convergence of $S_n(f, x_0)$ for x_0 in $(-\pi, \pi)$ and f in $L^1[-\pi, \pi]$ with period 2π. [1 + 1]

(f) If $g \in L^1[-\pi, \pi]$, has period 2π and differentiable at x_0, for x_0 in $(-\pi, \pi)$ show that $S_n(g, x_0)$ is convergent. (1)

(g) State Jordan's condition for convergence of $S_n(k, x_0)$ for $k \in L^1(-\pi, \pi)$ with period 2π. (1)

(h) Let $q : [-\pi, \pi] \to \mathbb{C}$ be continuous, periodic with period 2π. State and prove Fejer's theorem for $\sigma_n(q)$. [1 + 3]

2. Let $f \in L^1(\mathbb{R})$, $\hat{f} \in L^1(\mathbb{R})$ and f continuous. Let μ be a complex valued Borel measure on R. Find a relation between

$$\int f(x)\ d\mu(x) \quad \text{and} \quad \int \hat{f}(\xi)\ \ \hat{\mu}(-\xi)\ d\xi$$

and prove your claim. [Note that Fubini's theorem is valid for positive measures; if you are using Fubinis theorem for complex measures justify it]. [1 + 3]

3. a) Let $f \in L^1(R^n)$. Show that given $\epsilon > 0$, there exists h in $L^1(R^n)$ such that $\|h\| \le \epsilon$ and $\hat{h}(s) = \hat{f}(0) - \hat{h}(s)$ in a neighbourhood $N(\epsilon)$ of 0. [6]

(Hint: The family h_λ given for $\lambda > 0$ by $h_\lambda(x) = \hat{f}(0)\ g_\lambda(x) - (f * g_\lambda)(x)$, with $g_\lambda(x) = \frac{1}{\lambda^n}\ g(\frac{x}{\lambda})$ for a suitable g may help).

(b) Let $0 < t < s$. Define $g = \mathcal{X}_{[0,t]} + \mathcal{X}_{[0,s]}$. If s/t is irrational show that the linear span of the translates of g is a dense linear subspace of $L'(R)$. [3]

4. Let $a : R \to \mathbb{C}$ be a function. a is called a L^∞ atom if there is an interval $I \subset R$, such that (i) supp $a \subset I$, (ii) $\|a\|_\infty \le \frac{1}{\text{length}_I}$ and (iii) $\int\limits_R a = 0$. It is known that for the Hilbert transform H we have

$$\sup\{\|Ha\|_{L^1(R)} : a \text{ is an atom}\} < \infty.$$

Let $f : R \to \mathbb{C}$ be a measurable function such that $|f(x)| \le \frac{C}{(1+|x|)^{1+\epsilon}}$ for some $\epsilon > 0$ and some constant C and $\int f = 0$. Show that Hf is in $L'(R)$.

Hint: Write $f : \sum\limits_1^\infty \lambda_j a_j$ where a_j are L^∞ atoms and $\sum |\lambda_j| < \infty$. For properties of atom see above. [7]

5. Let $f \in L^2(R)$ with supp $f \subset [0, \infty)$. Define $G(x+iy)$ for x in R, $y < 0$ by

$$G(x+iy) = \frac{1}{\sqrt{2\pi}} \int dt\ f(t)\ e^{-it(x+iy)}.$$

Show that

(i) G is analytic in the set [3]

$$L = \{x + iy \in \mathbb{C} : y < 0\}.$$

(ii) $\int\ |G(x+iy) - \hat{f}(x)|^2\ dx \to 0$ as $y \to 0$. [1]

6. (a) Let $f : R \to \mathbb{C}$ be C^1 function such that $|f(x)| + |f'(x)| \leq K(1 + |x|)^{-1-\epsilon}$ for some $\epsilon > 0$ and some constant K. Show that

$$\sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \sum_{-\infty}^{\infty} \hat{f}(k)\ \frac{e^{ikx}}{\sqrt{2\pi}}.$$

[4]

7. (a) Let $\mathcal{S}(R)$ be the Schwartz class of functions on R. Define $Q, P : \mathcal{S}(R) \to \mathcal{S}(R)$ by $(Qf)(x) = x\ f(x)$ and $(Pf)(x) = -if'(x)$. Show that (i) $\langle Pg_1, g_2 \rangle = \langle g_1, Pg_2 \rangle$ and (ii) $\|f\|_2^2 \leq 2\ \|Qf\|_2 \|Pf\|_2$. [1 + 2]

(b) Let $f \in \mathcal{S}(R)$ and $\|f\|_2 = 1$. Define $u(f)$-uncertainty for f - by

$$u(f) = \{\|Qf\|^2 - \langle Qf, f \rangle^2\}^{\frac{1}{2}} \{\|Pf\|^2 - \langle Pf, f \rangle^2\}^{\frac{1}{2}}.$$

Let $k > 0$. Define g by $g(t) = \sqrt{k}\ f(kt)$. Find a relation between $u(f)$ and $u(g)$ and prove your claim. [4]

8. Let $f \in L^2(R)$. Assume that there exists constants $0 < A \leq B < \infty$ such that

(a)

$$\begin{aligned} A \sum_{k=-\infty}^{\infty} |a_k|^2 \ &\leq \ \int \left| \sum_k a_k f(t-k) \right|^2 \ dt \\ &\leq \ B \sum_k |a_k|^2 \end{aligned}$$

for all $a_k \in \mathbb{C}$.

Show that
$$\frac{A}{2\pi} \leq \sum_k \ |\hat{f}(\xi + 2k\pi)|^2 \leq \frac{B}{2\pi} \ ae \ \ \xi.$$

[4]

(b) Let $\{f_k : k \in Z\} \subset L^2(R)$ satisfy

$$\sum_k \ |a_k|^2 = \left\| \sum_k \ a_k f_k \right\|_{L^2}^2 \quad \text{for all } a_k \in \mathbb{C}.$$

Show that $\langle f_k, f_j \rangle = \delta_{kj}$. [2]

(c) Let f be as in (a). Let

$$L = \text{closed span } \{f(t-k) : k \text{ is in } Z\}.$$

Prove rigorously

$$\hat{L} = \left\{ a(\xi) \ \hat{f}(\xi) : \begin{array}{l} a \text{ has period } 2\pi, \\ a \in L^2[0, 2\pi] \end{array} \right\}$$

[3]